Cluster formation in the system of interacting Bose particles

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Based on statistical approach we described possible formation of spatially inhomogeneous distribution in the system of interacting Bose particles. The condition of cluster formation in both gas and condensed phases was obtained in this system. We studied the dynamics of cluster formation in the limit case of high temperatures. We compared the cluster-formation processes in the attractive system (with short-range interaction) and in the gravitational system at the low temperatures of Bose-Einstein condensate regime.

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I. INTRODUCTION

The formation of a spatially inhomogeneous distribution of interacting particles is a typical problem in condensed matter physics. The conditions for the formation of such structure are determined by the type of interaction. In the articles in [1,2] a new approach to a statistical description of interacting particles and phase transitions accompanied by cluster formation was proposed. A cluster is described by the function of the spatial distribution of particles. This function is a soliton solution of the nonlinear equation, which arises in most cases from a statistical description of interacting particles. A statistical description of the system of interacting particles developed in papers Refs. [1–3] is based on the application of the apparatus of quantum field theory [4–9,9,10] and gives the possibility to find the spatial distribution of particles, to calculate the cluster's size, and to determine the temperature of the phase transition into the state under consideration. By means of this description it has been shown that there is a possibility of cluster formation in the system of attracting particles. The basic equation for the function of the spatial distribution in the limit case of high temperatures—that is, in the case of the Boltzmann statistic—has been found. However, the dependence of equilibrium size of the cluster on thermodynamical conditions has not been determined, and the dynamics of its formation has not been considered. As this approach is correct for different statistics too, as well the task of describing such systems like a gas of interacting Fermi and Bose particles arises, where spatial inhomogeneity of the present type can appear.

Lately interest in the Bose condensate of particles with negative scattering length has increased. Experiments [11–13] have shown that the condensate collapses when the number of particles is sufficient. The same result is given by a numerical solution of the Gross-Pitaevskii equation [14]. The collapse is tunneling through barrier of attraction of particles and quantum pressure which is the consequence of wave packet diffusion. Such a situation is real if between the particles there is a short-range attracting potential that can be described by the scattering length. It is rather interesting to compare the stability of the model condensate with the long-range attracting potential—for example, $1/R$ (gravitating gas)—and the condensate with short-range attracting potential (described by negative scattering length) by testing the Gross-Pitaevskii equation solution of the stability. Perhaps, such a model can be useful for investigating the early stages of a dynamically changing universe [15].

In this article, based on a statistical approach [1–3], we describe the formation of a spatially inhomogeneous distribution in a system of interacting Bose particles. We obtain the conditions of cluster formation in both the Bose gas and condensate systems, and we describe the dynamics of cluster formation in the limit case of high temperatures (Boltzmann statistics). We compare the properties of spatial inhomogeneity in a Bose condensate of particles with negative scattering length and of particles with long-range attraction about their instability to collapse.

II. STATISTICAL APPROACH

Let us consider an interacting particle system being in such conditions when, on the one hand, the wave's thermal length of a particle can be larger than the average distance between them, so that it is necessary to take into account the type of statistic, but on the other hand, this length is by far smaller than the average scattering length that allows one to describe the interaction classically disregarding dynamical quantum correlations. The Hamiltonian of such a system [1–3,10,16] is

$$H(n)=\sum_{s}\varepsilon_{s}n_{s}-\frac{1}{2}\sum_{ss'}W_{ss'}n_{s}n_{s'}+\frac{1}{2}\sum_{ss'}U_{ss'}n_{s}n_{s'}, \qquad (1)$$

where ε_s is the additive part of the particle energy in the state s (for example, kinetic energy or energy in an external field), and $W_{ss'}$ and $U_{ss'}$ are the absolute values of the attraction and repulsion energies of particles in the states s and s', respectively. The macroscopic state of the system is determined by the occupations numbers n_s. The subscript s corresponds to variables that describe an individual particle state.

In Refs. [1–3] in order to investigate the thermodynamical properties of the interaction particle system a Hubbard-

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Stratonovich [17–20] representation has been used for the partition function:

$$Z_n = \frac{1}{2\pi i}\oint d\xi \int D\varphi \int D\psi \exp[-S(\xi,\varphi,\psi)]. \qquad (2)$$

The integral is calculated by the method of the "saddle point" [21,22] across the point $\partial S/\partial\xi=\partial S/\partial\varphi=\partial S/\partial\psi=0$. This equation provides a solution of a multiparticle problem in the sense that it selects those states of the system whose contributions to the partition function are dominant. S is the function which we call an action:

$$S(\xi,\varphi,\psi) = \frac{1}{2\beta}\sum_{s,\acute{s}} (W^{-1}_{s,s'}\varphi_s\varphi_{s'} + U^{-1}_{s,s'}\psi_s\psi_{s'}) + \sum_s \ln[1-\xi \times\exp(-\beta\varepsilon_s+\varphi_s)\cos\psi_s] + (N+1)\ln\xi, \qquad (3)$$

where ξ is activity, $\beta=1/\mathrm{kT}$ is reverse temperature, s and s' run all the states of the system, ε is the kinetic energy, and N is the number of particles. Two additional fields φ and ψ are introduced corresponding to attraction and repulsion. The partition function (3) is written as a functional integral over these fields. $W^{-1}_{s,s'}$ and $U^{-1}_{s,s'}$ are inverse operators of the interaction: $\omega^{-1}_{ss'}=\delta_{ss'}\hat{L}_{s'}$ where $\hat{L}_{s'}$ is such an operator for which the interaction's potential is a Green function. But it is a problem for each operator to find an inverse one. It can be done for the potential of hard spheres, screened Coulomb potential, and Newton potential [10]; that is why we will confine ourselves to gravitating Bose gas.

In the continuum approximation, the subscript s runs through a continuum of values in the system of volume V. When integrating over impulses and coordinates, we bear in mind that the unit of cell's volume in the space of individual states is equal to $\omega=(2\pi\hbar)^3$.

A. Gravitating Bose gas of hard spheres without a condensate

In this section we demonstrate the possibility of existence of a spatially inhomogeneous distribution in the system of interacting Bose particles, obtain the conditions of cluster formation in this system, and study the dynamics of cluster formation in the limit case of high temperature.

First of all let us consider a Bose gas of hard spheres of diameters a. The inverse operator of interaction looks as $U^{-1}_{rr'}=(1/U_0)\delta_{rr'}$ [1], where $U_0\to\infty$. Then, using Eqs. (3) and (A1), we have

$$S = -\frac{V-V_0}{\lambda^3}g_{5/2}(\xi) + \ln(1-\xi) + (N+1)\ln\xi + \frac{1}{\omega}\int_0^{V_0} dV\int_0^\infty d^3p \ln[1-\xi\exp(-\beta\varepsilon_p)\cos\psi], \qquad (4)$$

where $d^3p\equiv 4\pi p^2dp$ is the differential of volume in the space of impulses, $V_0\approx 2v_0N$ is the volume which will be occupied by the particles if they are collected close to each other, and $v_0=\frac{4}{3}\pi(a/2)^3$ is the volume of one particle. The equation for the saddle point is

$$\frac{\partial S}{\partial\psi} = \frac{V_0}{\omega}\int_0^\infty d^3p\frac{\xi\exp(-\beta\varepsilon_p)\sin\psi}{1-\xi\exp(-\beta\varepsilon_p)\cos\psi} = 0. \qquad (5)$$

The solution of this equation is

$$\psi = \begin{cases}\pi, & R<R_0,\\ 0, & R>R_0,\end{cases} \qquad (6)$$

where $R_0=\sqrt[3]{3V_0/4\pi}$. The solution (6) means that so far as two particles cannot come lesser distance than their diameters, then the system can not be compressed to a volume less than the volume V_0.

Now we shall consider a system of particles interacting by gravitational attraction and hard-sphere repulsion. For Newtonian attraction the inverse operator is known to be $W^{-1}_{rr'}=(-1/4\pi Gm^2)\Delta_r\delta_{rr'}$, where G is the gravitational constant, m is a particle's mass, and Δ_r is the Laplace operator. Then, using the result of the hard-sphere model (6), the action is written in the following way:

$$\begin{aligned}S &= \frac{1}{2\beta}\int_{V_0}^V \frac{(\nabla\varphi)^2}{4\pi Gm^2}dV + \frac{1}{\omega}\int_{V_0}^V dV\int d^3p \ln\left[1-\xi e^{\varphi}\times\exp\left(-\beta\frac{p^2}{2m}\right)\right] + \frac{1}{\omega}\int_0^{V_0}dV\int d^3p\ln\left[1+\xi\times\exp\left(-\beta\frac{p^2}{2m}\right)\right] + \ln(1-\xi) + (N+1)\ln\xi\\ &= \int_{V_0}^V dV\left[\frac{(\nabla\varphi)^2}{4r_m} - \frac{1}{\lambda^3}g_{5/2}(\xi e^{\varphi})\right] + \frac{V_0}{\lambda^3}f_{5/2}(\xi) + \ln(1-\xi) + (N+1)\ln\xi, \end{aligned} \qquad (7)$$

where $r_m=2\pi Gm^2\beta$, and it should be noted that a special Fermi function

$$f_{5/2}(\xi) = \frac{4}{\sqrt{\pi}}\int_0^\infty dx x^2\ln(1+\xi e^{-x^2}) = \sum_{l=1}^\infty(-1)^{l+1}\frac{\xi^l}{l^{5/2}}$$

arises [21,22]. It happens due to the repulsion of hard spheres which changes the behavior of Bose particles statistically.

Let us introduce the dimensionless quantity $r=R/r_m$ instead of R, a new variable $\sigma=\exp(\varphi/2)$, and mark in $\alpha^2\equiv r_m^3/\lambda^3$. Since there is a logarithm in expression (7) and $0<\xi<1$, the condition that is applied to the field σ is $0<\xi\sigma^2<1$.

Let us consider the case $\xi<1$, which corresponds to the absence of a Bose condensate: $\langle n_0\rangle/V\to 0$. The action (in spherical coordinates) (7) can be written in terms of a new above-mentioned variable:

$$S = 4\pi\int_{r_0}^\infty\left[\left(\frac{1}{\sigma}\frac{\partial\sigma}{\partial r}\right)^2 - \alpha^2 g_{5/2}(\xi\sigma^2)\right]r^2dr + \frac{V_0}{\lambda^3}f_{5/2}(\xi) + (N+1)\ln\xi. \qquad (8)$$

The equation for the saddle point is the equation of Lagrange for the functional (8). Here it is

$$\frac{\partial^2\sigma}{\partial r^2}-\frac{1}{\sigma}\left(\frac{\partial\sigma}{\partial r}\right)^2+\frac{1}{2}\alpha^2\frac{\partial g_{5/2}(\xi\sigma^2)}{\partial\sigma}\sigma^2 \equiv \frac{\partial^2\sigma}{\partial r^2}-\frac{1}{\sigma}\left(\frac{\partial\sigma}{\partial r}\right)^2+\alpha^2\sigma^2\sum_{l=1}^{\infty}\frac{\xi^l\sigma^{2l-1}}{l^{3/2}}=0. \quad (9)$$

This equation has not analytical solution. Let us consider the limit case $\xi\to 0$ (Boltzmann gas). Then (9) is reduced to

$$\frac{\partial^2\sigma}{\partial r^2}-\frac{1}{\sigma}\left(\frac{\partial\sigma}{\partial r}\right)^2+\xi\alpha^2\sigma^3=0. \quad (10)$$

This equation has a soliton solution [1]

$$\sigma=\frac{\Delta}{\sqrt{\xi}\alpha}\frac{1}{\cosh\Delta r}, \quad (11)$$

where Δ is an unknown integration constant which will be determined below. Any soliton solution corresponds to a spatially inhomogeneous distribution of particles—a finite-size cluster. The corresponding asymptotics for Eq. (11) are $\sigma^2=1$ for $r=d$, where d is the cluster size, and $\sigma\to 0$ as $r\to\infty$. This solution describes the presence of particles in the inhomogeneous formation of the size d and the absence of particles at infinity, since in this case the spatial distribution's function is

$$\rho(r)=m\xi\frac{1}{\lambda^3}\sigma^2, \quad (12)$$

with normalization $r_m^3\int\rho(r)d^3r=mN$. Let us substitute Eq. (11) in the action (8) taking into consideration that $\lim_{T\to\infty}f_{5/2}(\xi)=\xi$:

$$S=4\pi\int_{r_0}^{d}(\Delta^2-2\xi\alpha^2\sigma^2)r^2dr+\frac{V_0}{\lambda^3}\xi+(N+1)\ln\xi. \quad (13)$$

Then we will integrate using the decomposition $1/\cosh x\approx 1-x^2/2$ in power series of $x\equiv\Delta d\ll 1$:

$$S=-(V-V_0)\frac{\Delta^2}{\alpha^2\lambda^3}+\frac{V_0}{\lambda^3}i+(N+1)\ln\xi. \quad (14)$$

Δ^2 is found, from the asymptotic, that $1=(\Delta^2/\xi\alpha^2)[1-\Delta^2d^2]\Rightarrow\Delta^2\approx\xi\alpha^2+\xi^2d^2\alpha^4$. Thus, assuming that $V\ll V_0$, we have the result

$$S=-\frac{V-2V_0}{\lambda^3}\xi+(N+1)\ln\xi-\frac{V-V_0}{\lambda^3}\xi^2d^2\alpha^2, \quad (15)$$

where ξ is found from the saddle point equation $\partial S/\partial\xi=0$ assuming that $\lambda^3/V\gg\lambda^6/V^2$, $\xi_0\gg\xi_G$ [ξ_0 and ξ_G are activities of ideal (A2) and gravitating Bose gas correspondingly, and ξ_0^{sph} and ξ_G^{sph} are the same activities with the correction on the volume of the particles V_0]:

$$\begin{aligned}\xi&=\frac{\lambda^3(N+1)}{V-V_0}-\frac{2d^2\alpha^2\lambda^6(N+1)^2}{(V-V_0)^2}\\&\approx\frac{\lambda^3(N+1)}{V}+\frac{\lambda^3(N+1)V_0^2}{V^2}\\&\quad-\left(\frac{2d^2\alpha^2\lambda^6(N+1)^2}{V^2}+\frac{4d^2\alpha^2\lambda^6(N+1)^2V_0}{V^3}\right)\\&=\xi_0+\xi_0^{sph}+\xi_G+\xi_G^{sph}.\end{aligned} \quad (16)$$

Then integrating Eq. (15) on saddle point (16) in accordance with formula (2) we obtain the partition function

$$Z_N=Z_N^0\times\exp\left[\frac{V}{\lambda^3}(\xi_0^{sph}+\xi_G+\xi_G^{sph})-\frac{2V_0}{\lambda^3}\xi-(N+1)\times\ln\left(1+\frac{\xi_0^{sph}+\xi_G+\xi_G^{sph}}{\xi_0}\right)+\frac{V-V_0}{\lambda^3}\xi^2d^2\alpha^2\right], \quad (17)$$

where Z_N^0 is the partition function of ideal gas (A6). Knowing it we can find free energy of the system

$$F=F_0-kT\left[\frac{V}{\lambda^3}(\xi_0^{sph}+\xi_G+\xi_G^{sph})-\frac{2V_0}{\lambda^3}\xi-(N+1)\times\ln\left(1+\frac{\xi_0^{sph}+\xi_G+\xi_G^{sph}}{\xi_0}\right)+\frac{V-V_0}{\lambda^3}\xi^2d^2\alpha^2\right], \quad (18)$$

where F_0 is free energy of ideal Bose gas (A7). Minimizing Eq. (18) by the size of cluster $d=D/r_m$ and neglecting by the correction on the volume V_0 in gravitational part of the activity ξ_G^{sph} (since $\lambda^6V_0/V^3\ll\lambda^6/V^2\ll\lambda^3/V$),

$$\frac{\partial F}{\partial d}=-kT\frac{d\alpha^2\lambda^3 2(N+1)^2}{V-V_0}\times\left[1-4\frac{d^2\alpha^2\lambda^3(N+1)}{V-V_0}\right]=0, \quad (19)$$

we obtain the optimum radius of the cluster,

$$d_0^2=\frac{V-V_0}{4N\lambda^3\alpha^2}=\frac{V}{4Nr_m^3}\left(1-\frac{V_0}{V}\right) \quad (20)$$

or, in the dimension values,

$$D_0^2=\frac{1}{4}\frac{VkT}{2\pi Gm^2N}\left(1-\frac{V_0}{V}\right). \quad (21)$$

The decrease of the cluster's size with the increase of the number of particles in the system N is connected with a closer packing of the particles in the cluster on account of the increase of the gravitational energy. The rising of the cluster's size with temperature is connected with less close packing of the particles in connection with the resist of thermal motion's energy to gravitational energy (Fig. 1). Such a situation is realized due to the long-range attraction $(1/R)$ of gravitational interactions.

Let us consider the dynamics of cluster formation. For it we will use the equation of motion as

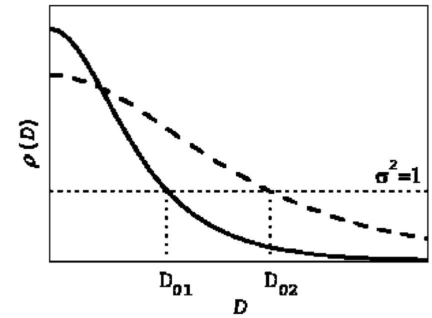

FIG. 1. The density distribution $\rho(D)$ of a cluster at different temperatures and quantities of particles (schematically). The solid line corresponds to a lower temperature (T_1) or greater particle number (N_1): the dashed line corresponds to a higher temperature (T_2) or smaller particle number (N_2): $T_2 > T_1, N_2 < N_1$. The dotted line $\sigma^2 = 1$ determines the equilibrium radii of clusters D_{01} and D_{02} under the above-mentioned thermodynamical conditions.

$$\frac{\partial D}{\partial t} = -\chi \frac{\partial F}{\partial D}, \tag{22}$$

where χ is the coefficient of inverse dimension of diffusion flow of mass through the cross section. Applying Eqs. (19) and (21) we have

$$\frac{\partial D}{\partial t} = \frac{\chi NkT}{2D_0^4}(-D^3 + DD_0^2). \tag{23}$$

Let us mark in $\eta \equiv \chi NkT/2D_0^4$. Then we rewrite Eq. (23) in a more convenient form:

$$\dot{D} + \eta D^3 - D_0^2 \eta D = 0. \tag{24}$$

The solution of this equation on condition that the initial state of the system be spatially homogeneous and assuming that $D(0)/D_0 = 1/2$ (Fig. 2) is

$$D^2 = \frac{D_0^2}{1 + 3\exp(-2\eta D_0^2 t)}. \tag{25}$$

Analogical result (exponential approaching to the equilibrium size) has been obtained in paper [23].

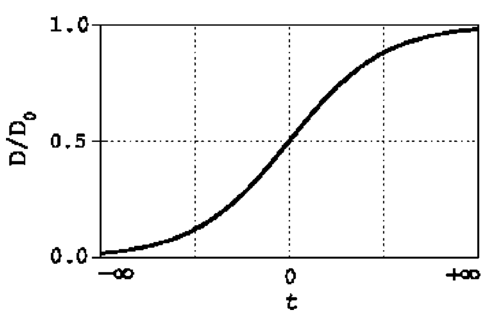

FIG. 2. The dynamics of cluster formation in a gravitating Bose gas (schematically). Here D/D_0 is a ratio of the cluster size in a certain moment of time to the equilibrium size. An accidentally $(t = -\infty)$ formed fluctuation of density in the system brings about the formation of a cluster. Its size D approaches the equilibrium size D_0 asymptotically $(t = +\infty)$.

Now let us investigate the asymptotics of the solution (25) for the stability. To do it let us consider a small deflection from Eq. (25),

$$X(t) = D(t) - D^0(t), \tag{26}$$

where

$$D^0(t) = \begin{cases} 0 \text{ on } & t \to -\infty, \\ D_0 \text{ on } & t \to +\infty. \end{cases}$$

$D^0(t) = 0$ means that the state of the system is spatially homogeneous; $D^0(t) = D_0$ means that the state is spatially inhomogeneous with the cluster of equilibrium size D_0. Let us mark in $B \equiv (\eta D_0^2 D - \eta D^3)$. Expanding $B(D)$ in power series of $X(t)$ and neglecting powers higher than the first we will obtain the equation for the small deflection:

$$\dot{X} = \frac{dB}{dD}_{D=D^0} X = [D_0^2 \eta - 3\eta (D^0)^2] X. \tag{27}$$

The solution of this equation is

$$X(t) \sim \begin{cases} \exp(D_0^2 \eta t) \text{ on } & D^0 = 0, \\ \exp(-2D_0^2 \eta t) \text{ on } & D^0 = D_0. \end{cases} \tag{28}$$

Suppose that the initial state of the system is spatially homogeneous $(D=0)$. It is not unstable because the small deflection (26) increases exponentially $\sim \exp(kt)$ [where k is the Liapunov index in Eq. (28)]. Some fluctuation of density that has appeared in the system brings about the appearance of the gradient of the gravitational potential. In its turn, it brings about the spatial inhomogeneity—a cluster with the size approaching to the equilibrium value (21) asymptotically. This spatially inhomogeneous state with the cluster of size $D = D_0$ is stable because the small deflection decreases exponentially $\sim \exp(-kt)$.

III. BOSE CONDENSATE

Let us suppose that we have two model Bose condensates. In one of them the particles interact by short-range attraction forces. Such an interaction is described by the scattering length $a < 0$ [24]. The other condensate consists of hard spheres with diameter $a_{sph} > 0$ and gravitational force acts between them only, which is long-range acting $1/R$. The gravitation interaction cannot be described by the scattering length, because the change of the phase of the S wave at scattering on the potential of effective radius r_0 is expressed as $-1/a + \frac{1}{2}k^2 r_0 + \cdots$ but for the Newton potential, $r_0 \to \infty$.

Condensates with negative scattering length were investigated in both experimental [11–13] and theoretical [14,25–30] works. It is proved to be that such the system becomes unstable to collapse if the number of atoms achieves a critical number N_c. Let us compare the properties of the Bose condensate of particles with negative scattering length and with long-range attraction about instability to collapse.

It is worth noting that the Bose condensate is a continuous wave of matter, a coherent state [31], and, hence, is described by some wave function, which is the product of one-particle functions in the first approximation [25]. On the other hand, the apparatus of statistical mechanics is based on the postulate of "accidental phases" [21], which is not carried out in the coherent state. That is why we cannot use the above-mentioned method for the investigation of spatial inhomogeneity in the condensed phase. Then for investigating such model systems we shall apply a method based on the Gross-Pitaevskii equation [24,32,33] (we are considering the spherical-symmetry problem only):

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial R^2} + \frac{2}{R}\frac{\partial\psi}{\partial R}\right) - \frac{4\pi\hbar^2|a|}{m}N|\psi|^2\psi + V\psi, \tag{29}$$

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi}{\partial R^2} + \frac{2}{R}\frac{\partial\varphi}{\partial R}\right) + \frac{4\pi\hbar^2 a_{sph}}{m}N|\varphi|^2\varphi + U\varphi. \tag{30}$$

Here ψ and φ are the wave functions of the condensates with densities $\rho(R)=mN|\psi|^2$ or $\rho(R)=mN|\varphi|^2$, and m is the mass of a particle. V is the energy of a particle in an external field (harmonic potential of the trap):

$$V = m\omega^2R^2/2. \tag{31}$$

U is the energy of a particle in the gravitation field of the condensate's mass distributed by low $\rho(R)$:

$$U = -mG\int\frac{\rho 4\pi R^2 dR}{R}. \tag{32}$$

This field plays the role of a field of the trap V due to the property of the long-range action.

Equation (29) is a nonlinear differential equation of the second order with variable coefficients. Equation (30) is a nonlinear integral-differential equation of the second order with variable coefficients. Hence, it is necessary to solve them numerically. These equations have soliton solutions under certain conditions that will be found out below. As in the previous section the availability of such solutions means the spatial inhomogeneity of the system cluster. The stability of the soliton solution of Eq. (29) has been investigated numerically in [14]. We will study and compare some general properties of the stability of the soliton solution of Eqs. (29) and (30) based on the equation for energy balance.

Let us assume that the condensate can be characterized by the mean density $\rho=mN/V$, where $V=(4\pi/3)(L/2)^3$ is a volume of the system and L is a spatial area occupied by the condensate. Then the potential energy of the condensate which described by Eq. (29) is as follows:

$$W = \frac{\hbar^2N}{2mL^2} - \frac{4\pi|a|\hbar^2}{mV}N^2 + \frac{3}{40}mN\omega^2L^2, \tag{33}$$

where the first addendum is the energy of quantum pressure [25], caused by the principle of indefiniteness. This energy resists compression of the gas. The second addendum is energy caused by a pseudopotential [21]. This energy aspires to compress the gas.

The third addendum is the energy of the condensate in the external field (31). This expression is obtained in the following way: $\int_0^{L/2}\rho\varphi dV$, where $\varphi=\omega^2R^2/2$ is potential of field of the trap, and $dV=4\pi R^2dR$ is differential of volume.

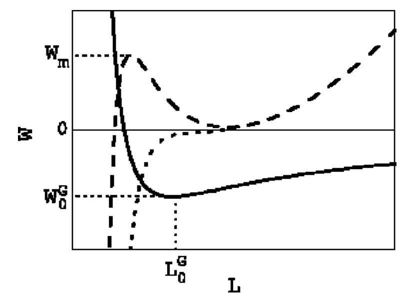

FIG. 3. The schematic dependence of the potential energy W of the Bose condensate on the value of the spatial area L occupied by the trapped condensate with short-range attraction (dash line) of particles and the gravitating Bose condensate (solid line). The first of them can be in the stable state, but it can collapse through the barrier by height W_m when the number of particles $N<N_c$, where N_c is a critical number. The condensate is sure to collapse when the number of particles is sufficient: $N>N_c$ (dotted line). The second of them is absolutely stable in the area L_0^G with energy W_0^G.

One can see in Fig. 3 that this Bose condensate is not stable. The instability to collapse happens due to the tunneling through the barrier of attracting the particles and quantum pressure. Let us evaluate the length and height of the barrier. It makes no sense to find the exact values because expression (33) is approximate and its derivation is an equation of the fifth order. The length of the barrier is

$$l \sim \sqrt{\frac{\hbar}{m\omega}} - |a|N. \tag{34}$$

The height of the barrier is

$$W_m \sim \frac{\hbar^2}{|a|^2Nm}. \tag{35}$$

From the formula (34) one can see that the barrier disappears (dotted line in Fig. 3) when the number of the particles is more than a critical number:

$$N_c \sim \frac{\sqrt{\hbar/m\omega}}{|a|}. \tag{36}$$

Let us appraise the region occupied by the model gravitating Bose condensate and its energy. The potential energy of this condensate which is described by Eq. (30) is as follows:

$$W = \frac{\hbar^2N}{2mL^2} + \frac{4\pi a_{sph}\hbar^2}{mV}N^2 - \frac{9}{10}G(mN)^2\frac{1}{L}, \tag{37}$$

where the first addendum is the energy of quantum pressure [25]. The second addendum is energy caused by a pseudopotential [21]. Since $a_{sph}>0$, then this energy aspires to widen the gas.

The third addendum is the energy of the condensate in a gravitational field of the condensate's mass. This expression is obtained in the following way: $\frac{1}{2}\int_0^{L/2}\rho\varphi dV$, where $\varphi = -G\int \rho dV/R$ is the gravitational potential of the field of the condensate's mass.

This energy has a minimum (Fig. 3)

$$W_0^G \sim -\frac{m^5N^3G^2}{\hbar^2} \tag{38}$$

at the point

$$L_0^G \sim \frac{\hbar^2}{m^3NG}\left(1+\sqrt{1+\frac{N^2a_{sph}Gm^3}{\hbar^2}}\right). \tag{39}$$

One can see in Fig. 3 that the gravitating Bose condensate is stable (it cannot collapse). Moreover, it is not in need of a trap, unlike the condensate with short-range attraction. Such behavior is the result of the long-range action of gravitation.

IV. CONCLUSION

In this paper we have studied the properties of the model system of gravitating Bose gas in two cases: for the area of above the point of a condensation, in particular within the Boltzmann limit, based on the new method given in papers Refs. [1–3], and in the area of under the point of condensation (near absolute zero). Our results are as follows.

The gravitational interaction of particles results in the formation of a cluster of finite size, as the initial homogeneous state is unstable. The size is determined by thermodynamical conditions; in particular, with raising the temperature the cluster size enlarges and it causes a decrease of the mean density; the cluster size decreases when particles are added in the system that is connected with the increase of the gravitational energy. Such behavior is the consequence of the long-range attraction of the gravitational interaction $(1/R)$. The size of the cluster approaches the equilibrium size asymptotically in the process of its formation. The state of the system with spatially inhomogeneous function of distribution corresponding to a cluster of equilibrium size (21) is stable.

The equilibrium state is spatially inhomogeneous in the gravitating Bose condensate. The gravitational interaction of the particles cannot be described in terms of the scattering length, because it is long-range acting. The comparison of properties of the condensate with negative scattering length to the model condensate with gravitational interaction (in Newton approach) has shown that unlike the first condensate, the second one is not in need of a trap and cannot collapse but takes an equilibrium size depending on the balance of the gravitation energy and quantum pressure energy.

Unfortunately, the results of the investigation of this model do not allow experimental verification, but they (cluster formations within the Boltzmann limit) can be useful for the problems of astrophysics, in the investigation of the formation of planet giants, stars, and their accumulation of gas-dust matter, in particular.

APPENDIX: IDEAL BOSE GAS

Let us obtain some expressions used in this paper for ideal Bose gas. Thus we will demonstrate the correctness and greater rationality of our approach as compared with the traditional method [21,22].

In this case $\varphi = \psi = 0$. Then the action (3) for the system is

$$S = \frac{1}{\omega}\int dV \int 4\pi p^2 dp \ln[1-\xi\exp(-\beta\varepsilon_p)] + (N+1)\ln\xi$$
$$= -\frac{V}{\lambda^3}g_{5/2}(\xi) + \ln(1-\xi) + (N+1)\ln\xi, \tag{A1}$$

where

$$g_{5/2}(\xi) = -\frac{4}{\sqrt{\pi}}\int_0^\infty dx x^2 \ln(1-\xi e^{-x^2}) = \sum_{l=1}^\infty \frac{\xi^l}{l^{5/2}},$$

special Bose function [21,22]. Hence it is clear that activity always $\xi < 1$ unlike the Fermi system; $\varepsilon_p = p^2/2m$ is the kinetic energy of the particles; $\ln(1-\xi)$ is the action for the condensed phase (the addendum with $\mathbf{p}=0$ is as important as the rest of the sum when $\xi \to 1$); $\lambda = \sqrt{\beta\hbar^2/2\pi m}$ is the wave's thermal length of a particle. Then the equation for the saddle point is as follows:

$$\frac{1}{v} = \frac{1}{\lambda^3}g_{3/2}(\xi) + \frac{1}{V}\frac{\xi}{1-\xi} \Leftrightarrow \frac{\lambda^3\langle n_0\rangle}{V} = \frac{\lambda^3}{v} - g_{3/2}(\xi), \tag{A2}$$

where $g_{3/2}(\xi) = \xi\partial g_{5/2}(\xi)/\partial\xi$, $v \equiv V/N$, and $\langle n_0\rangle$ is the occupation of the zero level. Then the value $\langle n_0\rangle/V$ is positive on condition

$$\frac{\lambda^3}{v} > g_{3/2}(1). \tag{A3}$$

Thus, we have a fallout of the terminal number of particles on the level with $p=0$ (so-called Bose-Einstein condensation). Part of the condensed particles is determined from expression (A2):

$$\frac{\langle n_0\rangle}{N} = 1 - \frac{v}{v_c} = 1 - \left(\frac{T}{T_c}\right)^{3/2}, \tag{A4}$$

with critical parameters

$$v_c = \frac{\lambda^3}{g_{3/2}(1)}, \quad T_c = \frac{2\pi\hbar^2}{[v g_{3/2}(1)]^{2/3}mk}. \tag{A5}$$

Let us consider the case $\xi < 1$. The occupation of the zero level is as insignificant as other levels with $p \neq 0$, which means the absence of a condensed phase. Such a situation is realized on condition $T > T_c$ and $v > v_c$. The partition function (2) is for this case

$$Z_N = \exp\left[\frac{V}{\lambda^3}g_{5/2}(\xi) - (N+1)\ln\xi\right], \tag{A6}$$

where ξ is determined from Eq. (A2) on condition of the condensed phase's absence. We can find thermodynamical functions knowing the partition function. The free energy and pressure of the system are

$$F = -kT\frac{V}{\lambda^3}g_{5/2}(\xi) + NkT\ln\xi, \tag{A7}$$

$$P = kT\frac{1}{\lambda^3}g_{5/2}(\xi). \tag{A8}$$

Let us consider the next case $\xi \to 0$ corresponding to high temperature (the Boltzmann limit). On this condition, expressions (A2) and (A6) are reduced to

$$Z_N = \exp\left[\frac{V}{\lambda^3}\xi - (N+1)\ln \xi\right] \approx \frac{V^N}{N!}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2N}, \tag{A9}$$

$$\frac{1}{v} = \frac{\xi}{\lambda^3}. \tag{A10}$$

At last, let us consider the case $\xi \to 1$. In this case $\langle n_0 \rangle / V$ is terminal and is determined by formula (A4). It means that the condensed phase is present in the system. Such a situation is realized at sufficiently low temperature and small volume $(T < T_c, v < v_c)$. Expression (A2) is reduced to

$$\frac{\xi}{1-\xi} \gg \frac{V}{\lambda^3}g_{3/2}(1) \Rightarrow N + 1 \approx \frac{\xi}{1-\xi}. \tag{A11}$$

In order to find the free energy in this case, let us define the partition function

$$Z_N = \exp\left[\frac{V}{\lambda^3}g_{5/2}(1) - \ln(1-\xi)\right]. \tag{A12}$$

Then,

$$\frac{F}{NkT} = -\frac{v}{\lambda^3}g_{5/2}(1), \tag{A13}$$

$$P = kT\frac{1}{\lambda^3}g_{5/2}(1). \tag{A14}$$

Let us find the internal energy U of the system proceeding from the arranged analogy between the apparatus of thermodynamics in our representation and the field theory. In order to do it let us use the correlation $-H = \partial S_{mech}/\partial t$ and determine the conformities $H \leftrightarrow U, S_{mech} \leftrightarrow S_{term}, t \leftrightarrow 1/kT$, where H is Hamilton's function of the system, S_{mech} and S_{term} are the actions for mechanic and thermodynamic (A1) systems, respectively, t is time, and $1/kT$ is reverse temperature. Then, with the help of expression (A1), we have

$$U = -\frac{\partial S}{\partial(1/kT)} = \frac{3}{2}\frac{VkT}{\lambda^3}g_{5/2}(\xi) = \frac{3}{2}PV. \tag{A15}$$

Expressions (A7)–(A9) and (A13)–(A15) coincide with the ones obtained by the usual way [21,22], and they it confirms the correctness of the proposed approach.

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